402. Proposed by C. N. SCHMALL, New York City.

If (x, y) be a double point on the curve $u \equiv f(x, y) = 0$, show that (1) the two branches of the curve will cut orthogonally if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0;$$

and (2), if this point be made the origin, then the equation of the tangents to the branches will be

$$(y'^2 - x'^2)\frac{\partial^2 u}{\partial x^2} + 2x'y'\frac{\partial^2 u}{\partial x \partial y} = 0$$

where (x', y') are the current coördinates of points on the tangents.

SOLUTION BY C. E. DIMICK, New London, Connecticut.

If (x, y) be a double point of the curve $u \equiv f(x, y) = 0$, $\partial u/\partial x = \partial u/\partial y = 0$ and the two values of dy/dx at the double point must satisfy the quadratic

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y}\frac{dy}{dx} + \frac{\partial^2 u}{\partial y^2}\left(\frac{dy}{dx}\right)^2 = 0.$$

(Todhunter's Diff. Calc., pages 319, 320.)

If the two tangents are perpendicular, the product of their slopes is -1, and since the product of the roots of the quadratic $a + bx + cx^2 = 0$ is a/c we have

$$\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial^2 u}{\partial y^2} = -1 \qquad \text{or} \qquad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

The equations of the two tangents will be $y - y_1 = l_1(x - x_1)$, and $y - y_1 = l_2(x - x_1)$, where l_1 and l_2 are the roots of the quadratic given above. Transposing and multiplying, the equation of the two tangents taken together will be

$$(y - y_1)^2 - (x - x_1)(y - y_1)(l_1 + l_2) + l_1 l_2 (x - x_1)^2 = 0,$$

which becomes on substituting the values of the sum and product of the roots of the quadratic

$$(y - y_1)^2 + 2(x - x_1)(y - y_1)\frac{\partial^2 u}{\partial x \partial y}\Big/\frac{\partial^2 u}{\partial y^2} + (x - x_1)^2\frac{\partial^2 u}{\partial x^2}\Big/\frac{\partial^2 u}{\partial y^2} = 0,$$

which, upon transforming to (x_1, y_1) as origin, becomes

$$y^2\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + x^2\frac{\partial^2 u}{\partial x^2} = 0.$$

But $\partial^2 u/\partial x^2 = -\partial^2 u/\partial y^2$ as the tangents are perpendicular.

Hence the equation reduces to

$$(y^2 - x^2)\frac{\partial^2 u}{\partial y^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} = 0,$$

the minus sign as given in the problem being incorrect.

Also solved by I. A. BARNETT, C. K. ROBBINS, GRACE M. BAREIS, J. A. BULLARD, F. M. MORGAN, and H. L. AGARD.

MECHANICS.

304. Proposed by B. F. FINKEL, Drury College.

A spherical shell, inner radius r and outer radius R, has within it a perfectly smooth solid sphere of the same material and with radius $r_1 < r$. If the inner surface of the spherical shell is also perfectly smooth, determine the motion, after the time t, of the shell and sphere down a rough inclined plane, inclination α.

SOLUTION BY H. S. UHLER, Yale University.

Let f denote the force exerted by the solid sphere of mass m_1 on the shell of radius of gyration k and of mass m. Also, let (x_1, y_1) and (x, R) be the coordinates of the centers (C_1 and C) of the solid and hollow bodies, respectively. f, x, x_1, and y_1 will be functions of the time t. By using the symbols k, m, and m_1 the equations given below apply to two cases in addition to the one proposed, namely, (i) a sphere within a cylindrical shell, and (ii) a cylinder inside a hollow cylinder. Moreover, in all three cases, the bodies are only required to have their centers of mass coincide with their geometric centers, so that the inner body may be hollow and all the bodies may be non-homogeneous radially, that is, the bodies may be built up of similar concentric or coaxial shells of different densities for the several layers. Neither does the density have to be a continuous function of the radius. In the given case $k^2 = 2(R^5 - r^5)/[5(R^3 - r^3)]$, $m = \frac{4}{3}\pi\delta(R^3 - r^3)$, and $m_1 = \frac{4}{3}\pi\delta r_1^3$, where δ symbolizes the constant density of the material involved.

Consider the outer body and take moments around the instantaneous axis through A. Then

$$m(k^2 + R^2)\frac{d^2\phi}{dt^2} = mgR \sin \alpha + fR \sin \theta,$$

where θ is the angle which the line $\overline{C_1C}$ makes at any time t with the fixed direction $\overline{AC}$ or $\overline{OY}$. The geometric meaning of ϕ is obvious from its defining equation $R\phi = x$ (no slipping of the body on the inclined plane). Then

$$\frac{d^2\phi}{dt^2} = \frac{1}{R}\cdot\frac{d^2x}{dt^2}$$

so that, writing s for $1 + k^2/R^2$,

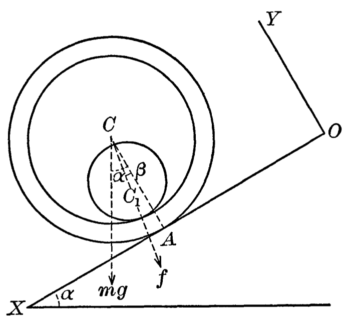
$$ms\frac{d^2x}{dt^2} = mg \sin \alpha + f \sin \theta. \tag{1}$$

The forces acting parallel to $\overline{OX}$ on the inner body fulfil the equation

$$m_1\frac{d^2x_1}{dt^2} = m_1g \sin \alpha - f \sin \theta. \tag{2}$$

The forces parallel to $\overline{OY}$ on this body satisfy the relation

$$m_1\frac{d^2y_1}{dt^2} = - m_1g \cos \alpha + f \cos \theta. \tag{3}$$



From the diagram, we see that

$$x_1 - x = \rho \sin \theta, \tag{4}$$

and

$$R - y_1 = \rho \cos \theta, \tag{5}$$

where $\rho \equiv r - r_1$, $(r > r_1)$.

Of the several differential equations that can be derived from the preceding five relations, the one connecting θ and t is the simplest.

Differentiating (4) and (5) twice each with respect to t we obtain, respectively,

$$\frac{d^2x_1}{dt^2} - \frac{d^2x}{dt^2} = \rho \cos \theta\frac{d^2\theta}{dt^2} - \rho \sin \theta\left(\frac{d\theta}{dt}\right)^2 \tag{4'}$$

and

$$\frac{d^2y_1}{dt^2} = \rho \sin \theta \frac{\theta d^2\theta}{dt^2} + \rho \cos \theta \left(\frac{d\theta}{dt}\right)^2. \tag{5'}$$

Multiplying equations (1), (2), (3), (4′), and (5′) by $m_1 \cos \theta$, $-ms \cos \theta$, $-(ms + m_1) \sin \theta$, $mm_1s \cos \theta$, and $m_1(ms + m_1) \sin \theta$, in the order named, and then adding, we get

$$\rho(ms + m_1 \sin^2 \theta) \frac{d^2\theta}{dt^2} + m_1\rho \sin \theta \cos \theta \left(\frac{d\theta}{dt}\right)^2 + g(ms + m_1) \cos \alpha \sin \theta - \frac{mgk^2}{R^2} \sin \alpha \cos \theta = 0. \tag{6}$$

Equation (6) may be reduced to the first order by observing that

$$2m_1 \sin \theta \cos \theta = \frac{d}{d\theta}(ms + m_1 \sin^2 \theta), \qquad \text{and} \qquad \frac{d^2\theta}{dt^2} = \frac{d\theta}{dt} \cdot \frac{d(d\theta/dt)}{d\theta}.$$

Accordingly

$$\rho(ms + m_1 \sin^2 \theta) \left(\frac{d\theta}{dt}\right)^2 - 2g(ms + m_1) \cos \alpha \cos \theta - \frac{2mgk^2}{R^2} \sin \alpha \sin \theta + c' = 0. \tag{7}$$

When $t = 0$, let $\theta = \theta_0$ and $d\theta/dt = \omega_0$, giving

$$c' = 2g(ms + m_1) \cos \alpha \cos \theta_0 + (2mgk^2/R^2) \sin \alpha \sin \theta_0 - \rho(ms + m_1 \sin^2 \theta_0)\omega_0^2.$$

Consequently,

$$t + f(\theta_0) = \int \frac{d\theta \sqrt{\rho(ms + m_1 \sin^2 \theta)}}{\sqrt{\begin{array}{r}\rho(ms + m_1 \sin^2 \theta_0)\omega_0^2 + 2g(ms + m_1) \cos \alpha (\cos \theta - \cos \theta_0) \\ - \dfrac{2mgk^2}{R^2} \sin \alpha (\sin \theta_0 - \sin \theta)\end{array}}}. \tag{8}$$

Since the analysis has been explicitly reduced to the evaluation of an indefinite integral the problem may be considered as formally solved. For, equation (8) theoretically gives θ as a function of t so that (5) then expresses the dependence of y_1 (and so also of dy_1/dt and d^2y_1/dt^2) upon t. Or, we may substitute directly in (5′) to get $d^2y_1/dt^2 = F(\theta) = \Phi(t)$. By suitably combining (1), (2), and (4′) it is ideally possible to exhibit x, x_1, dx/dt, dx_1/dt, d^2x/dt^2, and d^2x_1/dt^2 as explicit functions of t. However, since the writer does not know how to evaluate the integral in (8), save as a series development, he cannot profitably continue the general solution. Of course, the periodic nature and other properties of the function can be readily demonstrated but nothing is thereby added to what we know in advance from purely dynamical considerations.

Nevertheless, in the course of the investigation, we have noticed certain general and special facts which seem to be interesting. In the first place, elimination of f from (2) and (3) gives

$$\cos \theta \frac{d^2x_1}{dt^2} + \sin \theta \frac{d^2y_1}{dt^2} = g \sin (\alpha - \theta). \tag{9}$$

This equation shows that the sum of the components normal to $\overline{CC_1}$ of d^2x_1/dt^2 and d^2y_1/dt^2 is equal to the component in the same direction of the acceleration due to gravity, that is, the total linear acceleration of m_1 parallel to the common tangent of the inner and outer bodies is equal to the component of g in the same direction. This is a natural consequence of the perfect smoothness of the surfaces of contact. Relation (9) also indicates explicitly how d^2x_1/dt^2 can be obtained as a function of t as soon as θ and y_1 have been evaluated in terms of the time. Multiplying equations (4′) and (5′) in order by $\cos \theta$ and $\sin \theta$, and then subtracting the sum from (9) we find

$$\rho \frac{d^2\theta}{dt^2} + \cos \theta \frac{d^2x}{dt^2} = g \sin (\alpha - \theta). \tag{10}$$

This equation admits of the same physical interpretation as (9), relative motion of the bodies now being involved, and shows how to get d^2x/dt^2 directly as a function of the time.

As a second case, the result of adding equations (1) and (2) is

$$ms \frac{d^2x}{dt^2} + m_1 \frac{d^2x_1}{dt^2} = g(m + m_1) \sin \alpha. \tag{11}$$

If we take a point (ξ, η) which divides the line joining $C_1(x_1, y_1)$ to $C(x, R)$ internally in the ratio $ms : m_1$, then $\xi = (msx + m_1x_1)/(ms + m_1)$ so that (11) becomes

$$\frac{d^2\xi}{dt^2} = \frac{(m + m_1)g \sin \alpha}{ms + m_1}. \tag{12}$$

Therefore, this point, which is definitely associated with the moving system, has the property of constant acceleration parallel to the inclined plane. Equation (12) can be integrated at once giving $d\xi/dt$ and ξ as explicit functions of the time, quite independently of θ. On the other hand, $\eta = (msR + m_1y_1)/(ms + m_1)$; hence,

$$\frac{d^2\eta}{dt^2} = \frac{m_1}{ms + m_1} \cdot \frac{d^2y_1}{dt^2},$$

which is, in general, a function of θ.

An important special case of the motion of the system of bodies is brought out by (6). This equation is satisfied for all time if θ maintains the constant value θ' given by

$$\tan \theta' = \frac{mk^2}{mk^2 + (m + m_1)R^2} \cdot \tan \alpha; \tag{13}$$

for then $d^2\theta/dt^2 = d\theta/dt = 0$. Relation (13) shows that $\alpha > \theta' > 0$, as in the diagram. Under these conditions (4) and (5) give, respectively, $dx_1/dt = dx/dt$ and $dy_1/dt = 0$. Consequently, if we start with the centers of the inner and outer bodies at the respective points $(x_0 + \rho \sin \theta'$, $R - \rho \cos \theta')$ and (x_0, R), and impart equal linear velocities parallel to the incline, the inner body will not oscillate relative to its constraining wall but will maintain the constant angular position θ' while the shell slips under it. The equations of motion of the bodies are now easily shown to be

$$(ms + m_1)\frac{d^2x}{dt^2} = (ms + m_1)\frac{d^2x_1}{dt^2} = g(m + m_1) \sin \alpha,$$

$$(ms + m_1)\frac{dx}{dt} = (ms + m_1)\frac{dx_1}{dt} = g(m + m_1) \sin \alpha \cdot t + (ms + m_1)v_0,$$

$$(ms + m_1)x = \tfrac{1}{2}g(m + m_1) \sin \alpha \cdot t^2 + (ms + m_1)(v_0t + x_0),$$

$$(ms + m_1)x_1 = \tfrac{1}{2}g(m + m_1) \sin \alpha \cdot t^2 + (ms + m_1)(v_0t + x_0 + \rho \sin \theta').$$

Finally, if we introduce a new variable angle λ, defined by the equation $\lambda = \theta - \theta'$, and make use of (13) we can change the expression under the radical in the denominator of the integral in (8) to the form

$$\rho(ms + m_1 \sin^2 \theta_0)\omega_0^2 + 4g(ms + m_1) \cos \alpha \sec \theta' \sin \tfrac{1}{2}[(\theta' - \theta_0) + \lambda] \sin \tfrac{1}{2}[(\theta' - \theta_0) - \lambda],$$

which assumes the same value when λ is assigned values which are numerically equal but of opposite sign. Unfortunately for the analysis, the numerator of the integrand does not possess this kind of symmetry.

No other correct solution of this problem was received. Should any one integrate equation (8) we shall be glad to publish the result.—EDITORS.

NUMBER THEORY.

223. (October, 1914) Proposed by T. E. MASON, Purdue University.

Show that

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}}$$

is an integer, r, s, and t being positive integers. Generalize to the case of n integers, $r, s, t, u, \cdots$.

[Carmichael's *Theory of Numbers*, page 28.]

SOLUTION BY FRANK IRWIN, University of California.

Suppose we have rst objects, and let us divide them into t classes of rs objects each, then each class into s sub-classes of r objects each, and let us call each such classification, without any reference to order, a "classification" *par excellence*. We assert that the total number of such classifications is

$$\frac{(rst)!}{t!(s!)^t(r!)^{st}},$$

which expression is, consequently, an integer.